

The effects of strong correlations on the phase diagram and collective modes of quasi-two-dimensional dipolar fermions

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We calculate the ground state energy and phase diagram of the quasi-two-dimensional (quasi-2D) dipolar Fermi gas by variationally mapping the system to its strictly 2D counterpart for which accurate quantum Monte-Carlo (QMC) results are available. We further demonstrate the accuracy and flexibility of our variational technique by mapping the 2D dipolar Fermi gas to the 2D electron gas and showing that the resulting variational estimates closely match the QMC results. Our analysis suggests that the strongly-correlated Fermi liquid wavefunctions of different 2D Fermi systems share salient universal features. We show that inclusion of correlations significantly modify the dependence of the properties of the quasi-2D dipolar Fermi gas on the thickness of the layer compared to mean-field predictions, and calculate the frequency of its monopole oscillations in isotropic traps as an experimental demonstration of correlation effects.

Dipolar quantum gases have been the subject of much experimental and theoretical interest in the recent years. The long-range and anisotropic nature of dipole-dipole interactions can be exploited to realize a wide range of genuinely new quantum many-body phases [1]. Experiments with dipolar bosons have made rapid progress [2–4], with fermionic polar molecules and rare earth atoms emerging as particularly promising candidates for realizing strongly interacting dipolar Fermi gases owing to their large permanent dipole moments [5–9]. The collisional stability of trapped dipolar gases requires suppression of the attractive dipole-dipole (head-to-tail) scattering. This is achieved by tightly confining the gas in a thin “quasi-two-dimensional” layer and polarizing the dipoles perpendicular to it. The resulting effective dipole-dipole interaction is purely repulsive, has a long-range r^{-3} tail, and produces a Fermi liquid (FL) state with novel properties [10].

Although the theoretical investigation of dipolar fermions began more than a decade ago, the vast majority of obtained results rely on mean-field theory [11]. The natural next step is to investigate the role of correlations. Recently, Matveeva and Giorgini have carried out a fixed-node diffusion Monte-Carlo (FN-DMC) analysis of strictly 2D dipolar Fermi gas (2DDFG) at zero temperature and have shown a direct FL to triangular Wigner crystal (WC) transition in the strongly interacting regime. The mean-field theory predicts a density-wave instability as well [13], albeit at an interaction strength almost 20 times smaller than the QMC result.

In experiments with 2D dipolar fermions, the transverse confinement potential has a finite strength and the quasi-2D layer has a finite thickness as a result. The inclusion of this effect in realistic calculations is of particular importance here due to the strong anisotropy of dipole-dipole interactions. In this Letter, we use a simple variational method introduced earlier by two of the authors [14, 15] to incorporate the effect of finite transverse confinement in the ground state energy and the phase diagram of the quasi-2D dipolar Fermi gas (qDFG). This is achieved by using the variational principle

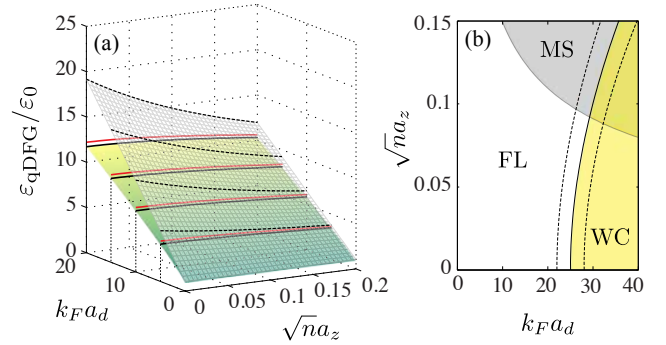


FIG. 1. (Color online) (a) The energy of quasi-2D dipolar Fermi gas (qDFG) as a function of dipole-dipole interaction strength $g_d \equiv k_F a_d$ (defined in the main text) and the layer width a_z in the units of $\varepsilon_0 \equiv \hbar^2 \pi n / m$. The upper surface and the accompanying black dashed lines show the result from Hartree-Fock theory while the lower surface and the black solid lines represent the variationally obtained result using 2DDFG ground state wavefunctions. The solid red lines show the variationally obtained energies using 2DEG ground state wavefunctions. The weak a_z -dependence of the variationally obtained results is due to correlations. (b) The variationally obtained phase diagram of qDFG. The MS region indicates the breakdown of the single-subband approximation (cf. main text for details). The dashed lines indicate the lower and upper uncertainty bounds for the WC transition line.

to approximately map the ground states of qDFG to a known system such as the two-dimensional electron gas (2DEG) or 2DDFG. In the first stage of the calculations, we assess the reliability of the variational technique by estimating the ground state energy and the WC transition point of 2DDFG using the ground states of the two-dimensional electron gas (2DEG) as trial wavefunctions. We show that the obtained variational estimates are remarkably close to the QMC results, capturing more than 95% of the correlation energy in the strongly interacting regime (see Fig. 2a) and predicting the correct location of the WC transition within the accuracy of QMC calculations. Although 2DEG and 2DDFG are

described by very different microscopic interactions and undergo crystallization in different regimes (low and high densities respectively), this analysis implies that these two seemingly different systems exhibit a very similar behavior in the strongly correlated regime even from a quantitative vantage point. In the second stage of calculations, we obtain the ground state energy and phase diagram of qDFG based on 2DDFG wavefunctions (see Fig. 1). We show that while mean-field theory predicts a strong dependence of the properties of qDFG on the transverse confinement width a_z , in actuality much of the short-ranged details of the effective interaction are masked by the exchange-correlation hole, resulting in a much weaker a_z -dependence. We finally discuss the experimental signatures of strong correlations in experiments with ultracold dipolar gases. We calculate the frequency of collective monopole (breathing) oscillations of qDFG in harmonic traps and show that correlations produce an observable effect on them.

The model. – We consider a gas of spinless dipolar fermions at zero temperature in an oblate and axially-symmetric confining potential $U_{\text{trap}}(\mathbf{r}) = m\omega_0^2(x^2 + y^2)/2 + m\omega_z^2 z^2/2$, such that $\omega_z \gg \omega_0$. We focus on the quasi-2D limit $\hbar\omega_z \gg \epsilon_F$ (ϵ_F is the Fermi energy) where only the lowest transverse subband is occupied. In this limit, the system can be described as a 2D Fermi gas with the following two-body interaction potential:

$$\mathcal{V}_{\text{dip}}^{\text{Q2D}}(\mathbf{r}) = \int dz dz' |\phi_0(z)|^2 |\phi_0(z')|^2 \mathcal{V}_{\text{dip}}^{\text{3D}}(\mathbf{r}, z - z'), \quad (1)$$

where $\mathcal{V}_{\text{dip}}^{\text{3D}}(\mathbf{r}, z - z') = D^2 (|\mathbf{r}|^2 - 3z^2)/|\mathbf{r}|^5$ is the dipole-dipole interaction in 3D space (D is the dipole moment and we have assumed that the dipoles are aligned along the z -axis) and $\phi_0(z) = e^{-z^2/(2a_z^2)}/(\sqrt{\pi}a_z)^{1/2}$ is the transverse wavefunction of particles in the lowest subband ($a_z \equiv [\hbar/(m\omega_z)]^{-1/2}$ is the transverse oscillator length and m is the mass of a single particle). The analytical expression for $\mathcal{V}_{\text{dip}}^{\text{Q2D}}(\mathbf{r})$ is given in the Supplemental Materials (SM) §1. We find that $\mathcal{V}_{\text{dip}}^{\text{Q2D}}(\mathbf{r}) \approx [2D^2/(\sqrt{2\pi}a_z^3)] \ln(a_z/r) + \mathcal{O}(1)$ for $r \ll a_z$ and $\mathcal{V}_{\text{dip}}^{\text{Q2D}}(\mathbf{r}) \approx D^2/r^3 - 9a_z^2/(2r^5) + \mathcal{O}(r^{-7})$ for large r . The single-subband limit is valid as long as $\mu < \hbar\omega_z$, where μ is the chemical potential and $\hbar\omega_z$ is the non-interacting band gap [19].

In the homogeneous case (no in-plane trap potential), the ground state can be labeled by two dimensionless parameters, $g_d \equiv k_F a_d$, and $\tilde{a}_z \equiv \sqrt{n}a_z$. Here, $k_F \equiv \sqrt{4\pi n}$ is the Fermi momentum, n is the 2D density and $a_d \equiv mD^2/\hbar^2$ is the “dipolar length”. Here, we confine our attention to the regime $\tilde{a}_z \ll 1$, where the single-subband approximation is applicable. In the presence of the trap potential, we treat the problem in the local density approximation (LDA) and parametrize the system with $\tilde{g}_d \equiv k_{F,0}a_d$ and $\tilde{a}_z \equiv \sqrt{n_0}a_z$, where $n_0 \equiv (2N)^{1/4}m\omega_0/(2\pi\hbar)$ is the density at the center of the trap for a non-interacting Fermi gas, and $k_{F,0} \equiv \sqrt{4\pi n_0}$.

The variational mapping method. – Variational methods

are known to work well if the class of trial states is sufficiently flexible. Having decent variational wavefunctions at one’s disposal can obviate time-consuming numerical calculations for many practical purposes. The states we consider here are the ground states of a repulsively interacting Fermi system, such as 2DEG or 2DDFG, whose ground states are accurately known. Depending on the strength of interactions, these states span a wide gamut, from a free Fermi gas to a strongly-correlated, crystal-like liquid. This choice guarantees that our variational energy estimate is asymptotically exact at both weak and strong coupling. We may therefore expect it to be rather accurate everywhere in between as well. In fact, in the intermediate strongly-correlated regime, the short-range details of interactions are masked by the localized exchange-correlation hole while the long-range tail is screened, making the wavefunctions of different systems bear strong resemblance to each other. The practical importance of this technique is the great simplicity and speed that it offers: by “recycling” what we already know about one strongly correlated system, we come up with decent variational estimates for another system at a small fraction of the computation time and effort that is usually required.

The procedure is formally described as follows: let $|\Psi_\xi\rangle^A$, $\varepsilon_K^A[n; \xi^A]$ and $g^A(r; \xi^A)$ be the normalized ground-state wavefunction, kinetic energy density and the pair-distribution function of a given system “A” at a fixed density n . Here, ξ^A is a dimensionless quantity that parameterizes the strength of interactions in “A” (e.g. in case of 2DEG, ξ can be taken as the Wigner-Seitz radius r_s). We propose $\{|\Psi_\xi\rangle^A\}$ as a family of variational wavefunctions for a second system “B”. Let $\varepsilon_K^B[n; \xi_{\text{var}}]$ and $\varepsilon_{\text{int}}^B[n; \xi_{\text{var}}]$ be the kinetic and interaction energy densities of “B” obtained by choosing $|\Psi_{\xi_{\text{var}}}\rangle^A$ as a trial wavefunction. Since the kinetic energy operator $\hat{K} \equiv -\sum_i \hbar^2 \nabla_i^2 / (2m)$ is identical for both systems, $\varepsilon_K^B[n; \xi_{\text{var}}] \equiv {}^A\langle \Psi_{\xi_{\text{var}}} | \hat{K} | \Psi_{\xi_{\text{var}}} \rangle^A = \varepsilon_K^A[n; \xi_{\text{var}}]$. The interaction energy density of “B” in the same trial state can be calculated using the “A” pair-distribution function:

$$\varepsilon_{\text{int}}^B[n; \xi_{\text{var}}] = \frac{n}{2} \int_0^\infty g^A(r; \xi_{\text{var}}) v^B(r; \xi^B) 2\pi r dr, \quad (2)$$

where $v^B(r; \xi^B)$ is the two-body interaction potential of “B” and ξ^B is a dimensionless measure of interaction strengths in “B”. Here we have also assumed, for concreteness, that both systems are spin-polarized 2D fermions with isotropic interactions. Minimizing $\varepsilon^B[n; \xi_{\text{var}}] \equiv \varepsilon_K^B[n; \xi_{\text{var}}] + \varepsilon_{\text{int}}^B[n; \xi_{\text{var}}]$ with respect to ξ_{var} , we obtain (1) a variational upper bound for the ground state energy of “B”, and (2) a mapping $\phi : \xi^B \rightarrow \xi^A$ that associates the ground states of “A” with the (approximate) ground states of “B”. The mapping also provides a heuristic method for obtaining the phase transitions of “B”: if a phase transition occurs at ξ_c^A for “A”, the mapping function predicts the same transition in “B” at $\phi^{-1}(\xi_c^A)$, provided that the inverse mapping exists. The accuracy of this simple scheme for predicting phase transition lies at the heart of our central premise, i.e. the universality of strongly-correlated wavefunctions near the FL-WC transition.

We refer to this scheme as *variational mapping*, and the results obtained by mapping to “A” as VM_A . Note that only the knowledge of $g^A(r; \xi^A)$ and $\varepsilon_K^A[n; \xi^A]$ is required here, and not the complete wavefunction. Furthermore, if the microscopic interaction is a power-law function, the kinetic energy density can be extracted from the total energy using the Virial theorem.

Variational mapping of 2DDFG to 2DEG.— In order to assess the accuracy of this method, we calculate the energy of the 2DDFG using the ground states of the 2DEG and compare to the FN-DMC results. Since our model here is composed of a single hyperfine state, we use ferromagnetic 2DEG states. We use the energies reported in Ref. [21] and the analytical representation of the 2DEG pair-distribution function provided in Ref. [22]. Fig. 2a shows the variationally obtained energies (green dashed lines) along with the FN-DMC result from Ref. [12] (solid lines). We find that the $\text{VM}_{2\text{DEG}}$ scheme remarkably captures more than 95% of the correlation energy of 2DDFG in the strongly interacting regime ($g_d > 20$). The transition from the ferromagnetic liquid to the WC phase occurs at $r_s = 28 \pm 3$ for the 2DEG [23] (r_s is the Wigner-Seitz radius), which variationally maps to $g_{d,\text{WC}}^{\text{var}} = 28 \pm 4$. This is again remarkably close to the FN-DMC prediction $g_{d,\text{WC}} = 25 \pm 3$ [12]. One can notice that the accuracy of the variational method improves for stronger interactions (see the inset plot of Fig. 2a). We associate this with the growth and localization of the exchange-correlation hole in the strongly correlated regime [24], which masks the short-range details of interactions better and also effectively screens its long-range tail. The difference between the exact and variationally obtained pair-distribution functions, shown in Fig. 2b (rightmost plots), is barely noticeable for large g_d .

Variational mapping of qDFG to 2DDFG.— Here, we take a much smaller step compared to the previous section and calculate the energy of the qDFG using the DFG ground states. We naturally expect the accuracy of the variational mapping procedure to be even better in this case, given that the 2D and quasi-2D effective dipole-dipole interactions have identical long-wavelength limits and only differ in their short-range details. We use the 2DDFG energies and pair-distribution functions reported in Ref. [12]. The details of the calculations are given in SM §2. Here, we also calculate the energy of qDFG using 2DEG ground states as a control variational estimate. Fig. 1a shows the obtained energy as a function of g_d and \bar{a}_z along with the Hartree-Fock result. The FL-WC phase diagram obtained from the mapping function is shown in Fig. 1b. We notice that $\text{VM}_{2\text{DEG}}$ (red lines) and VM_{DFG} (black lines, lower surface) energies are very close and follow the same trend consistently. We also find that $\text{VM}_{2\text{DEG}}$ -predicted energies are consistently slightly larger than VM_{DFG} energies, which simply indicates that DFG wavefunctions constitute a slightly better family of trial wavefunctions for qDFG. An interesting consequence of correlations is the essentially different dependence they produce for the ground state energy on

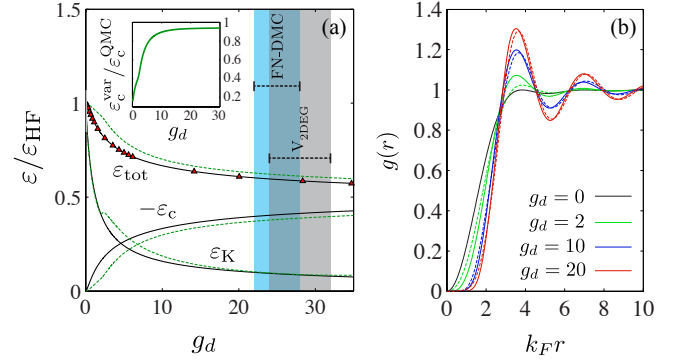


FIG. 2. (Color online) (a) The correlation (ε_c) kinetic (ε_{kin}), and total (ε_{tot}) energy densities of DFG in the units $\varepsilon_{\text{HF}} = (\hbar^2 \pi n / m)[1 + 128g_d / (45\pi)]$ (cf. SM §1 or Ref. [12]). The red triangles are the QMC results from Ref. [12] and the black solid lines are the analytical fits (provided in SM §2 for reference). The green dashed lines are our variational results based on 2DEG wavefunctions ($\text{VM}_{2\text{DEG}}$). The cyan and gray vertical bars represent WC transition as predicted by QMC ($g_d = 25 \pm 3$) and $\text{VM}_{2\text{DEG}}$ ($g_d = 29 \pm 4$) and coincide within the accuracy of the calculations. The inset plot shows the ratio of the correlation energy captured by $\text{VM}_{2\text{DEG}}$ and QMC as a function of g_d . (b) The pair-distribution function $g(r)$ of DFG obtained using QMC (solid lines) and $\text{VM}_{2\text{DEG}}$ (dashed lines). The black, green, blue and red lines correspond to $g_d = 0$ (Hartree-Fock), 2, 10, 20.

a_z , as compared to mean-field theory. Indeed, Hartree-Fock theory predicts a linear dependence of the total energy on a_z for small a_z (see the black dashed lines in Fig. 1a; the exact Hartree-Fock energy expression is also given in SM §1), while VM_{DFG} (and $\text{VM}_{2\text{DEG}}$) strongly suggest a quadratic a_z -dependence (see the solid black/red lines in Fig. 1a). In fact, we find that the VM_{DFG} energies can be parametrized to an excellent approximation as:

$$\varepsilon_{\text{qDFG}}(g_d, a_z) \approx \varepsilon_{\text{DFG}}(g_d) - \frac{\hbar^2 \pi n^2}{m} (c_0 + c_1 g_d) a_z^2, \quad (3)$$

in the parameter regime $5 < g_d < 30$ and $\sqrt{n} a_z < 0.1$. Here, $c_0 = 12.8$ and $c_1 = 1.45$ are obtained by fitting to the VM_{DFG} results, and $\varepsilon_{\text{DFG}}(g_d)$ is given in Ref. [12] and SM §2. The quadratic dependence of $\varepsilon_{\text{qDFG}}$ on a_z can be understood in simple terms: for large g_d , the short-range part of the effective dipole-dipole interaction is masked by the correlation hole and the a_z -dependence of the energy results from the leading correction to the interaction in the large- r limit, which is $\sim a_z^2 / r^5$ [cf. the discussion after Eq. (1)]. The FL-WC transition line can be parametrized as $g_{d,\text{WC}} \approx 25 + c_2 n a_z^2$, where $c_2 = 4.82 \times 10^2$. In contrast, the Hartree-Fock theory spuriously predicts a linear a_z -dependence for the FL stability phase boundary [13].

Experimental observation of correlation effects.— Among the important advantages of experiments with ultracold atoms and molecules as compared to traditional condensed matter systems are their cleanliness and their significantly slower dynamics. These features allow direct measurement of the equation of state [16] and the collective modes with a remarkably

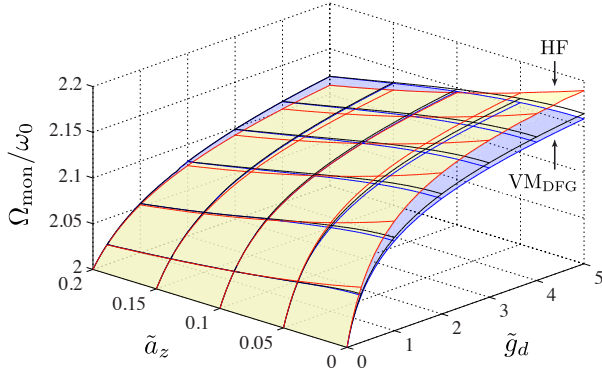


FIG. 3. (Color online) The monopole oscillation frequency Ω_{mon} of qDFG in an isotropic trap at $T = 0$. The yellow surface and red lines show the mean-field (Hartree-Fock) result. The blue surface and lines, and the black lines show the lower and upper bounds to Ω_{mon} once correlation are included using the VMDFG method. The bounds are obtained from hydrodynamic theory (lower) and sum rules. Inclusion of correlations changes the \tilde{a}_z -dependence of Ω_{mon} significantly.

high accuracy [17]. We showed that correlations have a profound effect on the properties of qDFG in the previous section. Here, we show that these effects can be directly observed by measuring collective oscillations of qDFG in optical or magnetic traps. Precise measurements of such have been successfully utilized to study the physics of the BEC-BCS crossover [17]. Here, we study qDFG in isotropic traps and calculate the frequency of monopole (breathing) oscillations, Ω_{mon} , about the equilibrium state.

Trapped qDFG. – The LDA equilibrium state of the trapped gas can be obtained from the balance of the trap restoring force and the internal pressure of the gas, i.e. $\partial_{\mathbf{r}} P[n_{\text{eq}}(\mathbf{r})] + n_{\text{eq}}(\mathbf{r}) \partial_{\mathbf{r}} U_{\text{trap}}(\mathbf{r}) = 0$. Here, $n_{\text{eq}}(\mathbf{r})$ is the local equilibrium density and $P[n]$ is the LDA pressure functional. At zero temperature, the pressure is given by $P[n] = \int_0^n dn' n' \partial(\mu[n'])/\partial n'$, where $\mu[n] = \partial(\epsilon[n])/ \partial n$ is the chemical potential. The equilibrium condition yields:

$$\frac{dn_{\text{eq}}}{dr} = -m\omega_0^2 \kappa[n_{\text{eq}}] n_{\text{eq}}^2 r, \quad (4)$$

where $\kappa[n] \equiv n^{-2}[\partial^2(\epsilon[n])/ \partial n^2]^{-1}$ is the compressibility, which can be calculated from the energies presented above. The equilibrium density profile is obtained by solving Eq. (4) under the global particle number constraint, $\int d^2\mathbf{r} n_{\text{eq}}(r) = N$. Correlations reduce the pressure at a given density, making the gas more compressible and resulting in a consistently smaller equilibrium radius of the trapped gas as compared to mean-field theory.

An exact treatment of collective oscillations at $T = 0$ requires solving the Landau kinetic equation, which is not feasible without the knowledge of the Landau parameters. Nevertheless, a lower bound to the frequency of collective oscillations can be found by resorting to the ideal hydrodynamic approximation. At zero temperature, the hydrodynamical description of the gas is given by the conservation laws for the

mass and momentum currents, i.e. $\partial_t n + \partial_{\mathbf{r}}(n\mathbf{v}) = 0$ and $m(\partial_t \mathbf{v} + \mathbf{v} \cdot \partial_{\mathbf{r}} \mathbf{v}) = -n^{-1} \partial_{\mathbf{r}} P - \partial_{\mathbf{r}} U_{\text{trap}}$, where \mathbf{v} is the macroscopic velocity field. Linearizing these equations about the equilibrium state and solving for $\delta n \equiv n - n_{\text{eq}}$ gives

$$\partial_t^2 \delta n + \partial_{\mathbf{r}} \cdot \left[n_{\text{eq}} \partial_{\mathbf{r}} \left(\frac{\delta n}{m \kappa[n_{\text{eq}}] n_{\text{eq}}^2} \right) \right] = 0. \quad (5)$$

The absence of dynamical Fermi surface deformations in the hydrodynamic approximation (which strictly increase the sound velocity in the case of isotropic repulsive interactions), implies that the obtained collective excitation energies are strictly lower than the exact values [25–27]. On the other hand, a strict upper bound is given by $(\hbar \Omega_{\text{mon}})^2 \leq m_{\mathcal{M}}^{(3)}/m_{\mathcal{M}}^{(1)}$, where $m_{\mathcal{M}}^{(1)}$ and $m_{\mathcal{M}}^{(3)}$ are the first and third moments of the monopole response function and can be evaluated using sum rules [28]. The numerical method for solving Eq. (5) and calculating the required moments are discussed in detail in SM §3. The mean-field analysis of collective modes in traps has appeared in several previous works [11, 18]. Here, we obtain Ω_{mon} by solving the Boltzmann-Vlasov equation using the numerically exact method given in Ref. [18].

Fig. 3 shows the obtained results. We note that the different a_z -dependence of the mean-field vs. correlated theory also persists in Ω_{mon} . For small values of \tilde{a}_z , correlations have a tendency to decrease the frequency of oscillations as compared to the prediction of the mean-field theory. This scenario is reversed, however, as \tilde{a}_z is increased. One can understand this reversal as a consequence of the weak dependence of the correlated theory on the width of the layer and the erroneously strong dependence of the mean-field theory. We note that while strictly speaking the results presented here correspond to the $T = 0$ limit, the correlation effects are expected to persist as long as $T < T_F$. In particular, the quadratic dependence of Ω_{mon} on the width of the layer only relies on the presence of the correlation hole and is expected to persist up to $T \sim T_F$. Suitably large dipolar interactions for observing such strong correlation effects are expected to be achievable in experiments with fermionic polar molecules such as KRb [5] and NaK [9].

Summary and outlook. – We have explored the idea of variational mapping between the ground states of different Fermi systems as a simple method for obtaining ground state energies of strongly correlated fermionic systems. We have found strong quantitative evidence for the accuracy of this simple variational scheme, which suggests a universality in strongly-correlated Fermi liquid wavefunctions. We have calculated the energy and the phase diagram of qDFG using this method and shown that correlations result in very significant corrections to the mean-field picture. Finally, we have calculated the frequency of monopole oscillations of qDFG as a simple experimental proposal for observing the correlation effects. As a final remark, we note that the variational mapping method may be extended to superfluid and multi-component Fermi gases, which might be useful in

investigating magnetism and superconductivity in electronic systems using dipolar gases as an experimental test-bed.

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SUPPLEMENTAL MATERIALS

1. The effective quasi-two-dimensional DDI

The formal expression for $\mathcal{V}_{\text{dip}}^{\text{Q2D}}(\mathbf{r})$ was given in the main text as an integral. The integration, however, can be done analytically and we find $\mathcal{V}_{\text{dip}}^{\text{2D}}(r) = [D^2/(2\sqrt{2\pi}a_z^3)]v(r/a_z)$, where:

$$v(x) = e^{x^2/4} [(2+x^2)K_0(x^2/4) - x^2K_1(x^2/4)], \quad (6)$$

and $K_n(x)$ denotes the modified Bessel function of the second kind of order n . For small x , we find:

$$v(x) = -4 - 2\gamma - 2\ln(x^2/8) + \mathcal{O}(x^2), \quad (7)$$

while for large x , we obtain the asymptotic expansion:

$$v(x) = 2\sqrt{2\pi} \left[\frac{1}{x^3} - \frac{9}{2x^5} + \mathcal{O}(x^{-7}) \right]. \quad (8)$$

The Hartree-Fock energy density at $T = 0$ can be easily calculated as:

$$\begin{aligned} \varepsilon^{\text{HF}} &= \varepsilon_0 + \frac{1}{4} \int_0^\infty x \left(1 - \frac{4J_1(x)^2}{x^2} \right) \mathcal{V}_{\text{dip}}^{\text{2D}} \left(\frac{x}{\sqrt{4\pi n}} \right) \\ &= \varepsilon_0 \left[1 + \frac{128}{45\pi} g_d I(\bar{a}_z) \right], \end{aligned} \quad (9)$$

where $\varepsilon_0 = \hbar^2 \pi n / m$ is the kinetic energy of a non-interacting spin-polarized 2D Fermi gas, and g_d and \bar{a}_z were defined in the main text, and $I(\bar{a}_z)$ can be expressed in terms of special functions:

$$\begin{aligned} I(\bar{a}_z) &= \frac{5}{2} {}_2F_2 \left(\frac{3}{2}, 2; \frac{5}{2}, \frac{5}{2}; 8\pi\bar{a}_z^2 \right) \\ &+ \frac{45}{8192\pi^2\bar{a}_z^5} \left[8\sqrt{2\pi}\bar{a}_z^2 \left(\text{Ei}(8\pi\bar{a}_z^2) - \log(8\pi\bar{a}_z^2) \right) \right. \\ &- \sqrt{2} (16\pi\bar{a}_z^2 - 3) e^{8\pi\bar{a}_z^2} \text{Erf}(\sqrt{8\pi}\bar{a}_z) + 32\sqrt{2}\pi^2\bar{a}_z^4 \\ &\left. - 8\sqrt{2}(\gamma - 3)\pi\bar{a}_z^2 - 24\bar{a}_z - 3\sqrt{2}(e^{8\pi\bar{a}_z^2} - 1) \right]. \end{aligned} \quad (10)$$

Expanding $I(\bar{a}_z)$ for small \bar{a}_z , we get:

$$\varepsilon^{\text{HF}}/\varepsilon_0 = 1 + \frac{128g_d}{45\pi} - 2\sqrt{2}g_d\bar{a}_z + \mathcal{O}(g_d\bar{a}_z^2). \quad (11)$$

2. Remarks on the application of the variational mapping method to 2DDFG

The application of the variational mapping method to 2DDFG is similar to mapping to 2DEG, as described in Ref. [1]. As mentioned in the main text, the only required information about the reference system are (1) the kinetic energy density, and (2) the pair-distribution function. As a first

step, we parametrize the QMC energies of 2DDFG reported in Ref. [2]. Motivated by the perturbative analysis of 2DDFG given in Ref. [3], and qualitative expectations about the dependence of the energy on g_d , we parametrize the 2DDFG correlation energy using the following ansatz:

$$\begin{aligned} \tilde{\varepsilon}_c(g_d) &= (Ag_d^2 + Bg_d^3 + Cg_d^4) \\ &\times \log \left[1 + \frac{1}{c_1g_d + c_{3/2}g_d^{3/2} + c_2g_d^2 + c_3g_d^3} \right]. \end{aligned} \quad (12)$$

Here, $\tilde{\varepsilon}_c(g_d) \equiv \varepsilon_c(g_d)/\varepsilon_0$, where $\varepsilon_0 = \hbar^2 \pi n / m$ is the kinetic energy of the non-interacting system. In the limit $g_d \ll 1$, the above ansatz gives $\varepsilon_c \approx -Ag_d^2 \log(c_1g_d) + \mathcal{O}(g_d^{5/2})$, which is in agreement with the perturbation analysis of Lu and Shlyapnikov [3], who find $A = -1/4$ and $c_1 \approx 1.43$. Here, we do not fix these parameters using the analytic solution and keep both as fitting parameters. Although we are only concerned with the liquid phase, we remark that the above ansatz also assumes the correct expected form $\sim \alpha/\sqrt{g_d} + \beta + \gamma g_d$ in the crystal phase [2]. Optimizing the parameters in the range $g_d \in [0, 28]$, we find $A = -0.2511$, $B = -0.192$, $C = -0.01093$, $c_1 = 1.492$, $c_{3/2} = 0.004919$, $c_2 = 0.5814$ and $c_3 = 0.02382$. Note that the optimized values of A and c_1 are very close to the known analytical values. The fit is clearly excellent. The RMSE is 5×10^{-4} .

In order to extract the kinetic energy density of 2DDFG from its total energy density, we use the Virial theorem. The general Virial theorem due to Cottrell and Paterson [4] implies $2\varepsilon_K - \frac{1}{N} \langle \sum_{i=1}^N \mathbf{r}_i \cdot \partial \hat{\mathcal{V}} / \partial \mathbf{r}_i \rangle + \ell \partial \varepsilon / \partial \ell = 0$, where $\hat{\mathcal{V}}$ is the two-body interaction operator, N is the number of particles and ℓ is a linear dimension of the system size. Strictly in 2D, DDI is a homogeneous function of order -3 and the Virial term evaluates to $3\varepsilon_{\text{int}}$. Noting that $g_d \propto \ell^{-1}$, we get $2\varepsilon_K + 3\varepsilon_{\text{int}} = 2\varepsilon + g_d \partial_{g_d} \varepsilon[g_d]$. Using $\varepsilon = \varepsilon_K + \varepsilon_{\text{int}}$, we finally get:

$$\varepsilon_K = \varepsilon - g_d \partial_{g_d} \varepsilon(g_d) = \varepsilon_0 [1 - g_d \partial_{g_d} \varepsilon_c(g_d)]. \quad (13)$$

The pair-distribution function of 2DDFG is reported in Ref. [2] for several values of g_d , using which we constructed an analytical representation for it in the spirit of the work by Gori-Giorgi *et al.* [5]. We will report the technical details of this procedure in a separate work.

3. qDFG in isotropic traps

3.1. The equilibrium density profile

The equilibrium density of the trapped gas can be obtained by solving Eq. (6) in the main text. For a non-interacting gas, $\varepsilon[n] = \varepsilon_0 = \pi \hbar^2 n / m$ and we find:

$$n_{\text{eq},0}(r) = \frac{m^2 \omega_0^2}{4\pi \hbar^2} (R_{\text{TF},0}^2 - r^2), \quad (14)$$

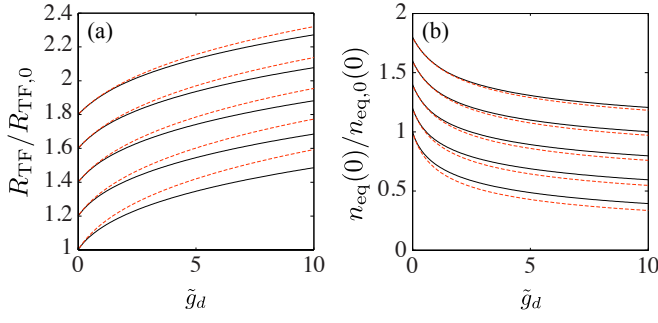


FIG. 4. The equilibrium radius (a) and density at the center of the trap (b) of qDFG at $T = 0$ as a function of \tilde{g}_d for various quasi-two-dimensionality factors (from bottom to top, $\sqrt{2\pi}\tilde{a}_z = 0, 0.1, 0.2, 0.3, 0.4$). The plots are shifted up in 0.2 increments for better visibility. The solid and dashed lines correspond to the correlated and mean-field results respectively. Note that $R_{\text{TF},0} = (8N)^{1/4}[\hbar/(m\omega_0)]^{1/2}$ and $n_{\text{eq},0}(0) = m\omega_0(2N)^{1/2}/(2\pi\hbar^2)$.

where $R_{\text{TF},0} = (2N)^{1/4}\sqrt{2}[\hbar/(m\omega_0)]^{1/2}$ is the Thomas-Fermi radius of the cloud and N is the number of particles. In the presence of interactions, the differential equation for n_{eq} may only be solved numerically. We arrive at a solution for n_{eq} by first finding upper and lower bounds to the density at center of the trap, n_0 using a direct search, and then solving for the function $n_{\text{eq}}(r)$ that satisfies the global particle number constraint, $\int d^2\mathbf{r} n_{\text{eq}}(r) = N$, by bisecting the bounding interval. Convergence within a relative error tolerance of 10^{-6} is achieved usually within 20 bisections.

Fig. 4 shows (a) the equilibrium radius of the cloud (b) the density at the center of the trap. The plots show both the results based on Hartree-Fock LDA energy density functional [Eq. 9] (red dashed lines) and the correlated energy obtained by variationally mapping to 2DDFG (black lines). Correlations make the gas more compressible and results in (1) larger density of particles at the center of the trap, and (2) smaller equilibrium radius.

3.2. Numerical solution of the linearized hydrodynamic equation

The linearized hydrodynamic equation for the density variations (Eq. 7 in the main text) can be solved as follows. First, we note that the equation can be put in a self-adjoint form by changing variables to $\Psi \equiv [m\kappa_{\text{eq}}n_{\text{eq}}^2]^{-1/2}\delta n$ [6]. We then obtain $(\partial_t^2 + \hat{\mathcal{G}})\Psi = 0$, where $\hat{\mathcal{G}} = -\mathcal{B}_{\text{eq}}\partial_{\mathbf{r}} \cdot \{n_{\text{eq}}\partial_{\mathbf{r}}[\mathcal{B}_{\text{eq}}\Psi]\}$ and $\mathcal{B}_{\text{eq}} = (m\kappa_{\text{eq}}n_{\text{eq}}^2)^{-1/2}$. It is easily verified that $\hat{\mathcal{G}}$ is self-adjoint with respect to the inner-product $\langle\Psi_1|\Psi_2\rangle = \int_{\Omega_+} d^2\mathbf{r} \Psi_1^*(\mathbf{r})\Psi_2(\mathbf{r})$, where $\Omega_+ = \{\mathbf{r} : n_{\text{eq}}(\mathbf{r}) > 0\}$. Moreover, $\hat{\mathcal{G}}$ is a Hermitian operator on the L^2 functions that remain finite on the boundary of Ω_+ .

In the absence of interactions, we can find the eigenfunctions of $\hat{\mathcal{G}}$ analytically. Here, we are interested in monopole oscillations and hence, only the zero angular momentum solutions are needed. The (unnormalized) eigenfunctions are

given by:

$$\Psi_n^{(0)}(r) = {}_2F_1(-n, n+1, 1, r^2/R_{\text{TF},0}^2), \quad (15)$$

corresponding to the eigenvalue $\nu_n = \omega_0\sqrt{2n(n+1)}$, where $n \in \mathbb{N}$. In the presence of interactions, the eigenfunctions can no longer be found analytically. We find the spectrum of $\hat{\mathcal{G}}$ by calculating the matrix elements of $\hat{\mathcal{G}}$ in the non-interacting basis and diagonalizing it. The spectrum consists of a single unphysical solution, $\Psi \sim \mathcal{B}_{\text{eq}}^{-1}$ with eigenvalue 0. The oscillation frequency of the lowest lying physical monopole mode, Ω_{mon} , corresponds to the smallest non-zero eigenvalue. We found that truncating the basis set to the first 10 non-interacting eigenfunctions yields Ω_{mon} to a relative accuracy level of 10^{-6} .

3.3. Evaluating moments of the monopole response function

We obtained the upper bound for Ω_{mon} using the exact relation between the moments of the response function, i.e. $(\hbar\Omega_{\text{mon}})^2 \leq m_{\mathcal{M}}^{(3)}/m_{\mathcal{M}}^{(1)}$. The monopole excitation operator and its corresponding response functions are $\hat{\mathcal{M}} = \sum_{i=1}^N r_i^2$ and $\chi_{\mathcal{M}}(\omega) = \hbar^{-1} \int dt e^{i\omega t} \langle [\hat{\mathcal{M}}(t), \hat{\mathcal{M}}(0)]_{\text{eq}} \rangle$ respectively. The first and third moments can be calculated using sum rules [7]:

$$\begin{aligned} m_{\mathcal{M}}^{(1)} &= \langle 0 | [[\hat{\mathcal{M}}, \hat{\mathcal{H}}], \hat{\mathcal{M}}] | 0 \rangle / 2, \\ m_{\mathcal{M}}^{(3)} &= \langle 0 | [[[[\hat{\mathcal{M}}, \hat{\mathcal{H}}], \hat{\mathcal{H}}], [\hat{\mathcal{H}}, \hat{\mathcal{M}}]] | 0 \rangle / 2, \end{aligned} \quad (16)$$

where $|0\rangle$ is the equilibrium state of the trapped gas. A direct calculation using commutation relations yield:

$$\frac{m_{\mathcal{M}}^{(3)}}{m_{\mathcal{M}}^{(1)}} = 2\omega_0 \left[\frac{4(\mathcal{E}_{\text{K}} + \mathcal{E}_{\text{trap}}) + \mathcal{E}_{\mathcal{V}_2}}{8\mathcal{E}_{\text{trap}}} \right]^{\frac{1}{2}}, \quad (17)$$

where:

$$\begin{aligned} \mathcal{E}_{\text{K}} &= \int d^2\mathbf{r} n_{\text{eq}}(\mathbf{r}) \varepsilon_{\text{K}}[n_{\text{eq}}(\mathbf{r})], \\ \mathcal{E}_{\text{trap}} &= \int d^2\mathbf{r} n_{\text{eq}}(\mathbf{r}) U_{\text{trap}}(\mathbf{r}), \\ \mathcal{E}_{\mathcal{V}_2} &= \int d^2\mathbf{r} n_{\text{eq}}(\mathbf{r}) \varepsilon_{\mathcal{V}_2}[n_{\text{eq}}(\mathbf{r})]. \end{aligned} \quad (18)$$

Here, $\varepsilon_{\mathcal{V}_2}[n]$ is given by:

$$\begin{aligned} \varepsilon_{\mathcal{V}_2}[n] &= \frac{\hbar^2\pi n}{m} \frac{1}{\sqrt{8\pi}} \frac{g_d[n]}{(4\pi)^{\frac{3}{2}}\bar{a}_z[n]^3} \\ &\times \int x g(x; g_d[n], \bar{a}_z[n]) v^{(2)}\left(\frac{x}{\sqrt{4\pi}\bar{a}_z[n]}\right). \end{aligned} \quad (19)$$

In the above equation, $v^{(2)}(x) = x\partial_x v(x) + x^2\partial_x^2 v(x)$, and $g_d[n]$ and $\bar{a}_z[n]$ are the local DDI strength and quasi-two-dimensionality. In Eq. (19), the pair-distribution function is that of the 2DDFG that variationally maps to $(g_d[n], \bar{a}_z[n])$.

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